

Central extensions of cotangent universal hierarchy: (2+1)-dimensional bi-Hamiltonian systems

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September 23, 2008

We introduce the cotangent universal hierarchy that extends the universal hierarchy from [1, 2, 3, 4, 5]. Then we construct a (2+1)-dimensional double central extension of the cotangent universal hierarchy and show that this extension is bi-Hamiltonian. This yields, as a byproduct, the central extension of the original universal hierarchy.

Keywords: cotangent universal hierarchy, central extension, integrable systems, (2+1)-dimensional bi-Hamiltonian systems, R -matrix

Introduction

The so-called universal hierarchy [1, 2, 3, 4, 5] is now a subject of intense research. This hierarchy is, in its original form, an infinite hierarchy of coupled dispersionless (1+1)-dimensional integrable systems. The universal hierarchy can be thought of as a model equation of soliton theory from which one can obtain many well-known and new soliton equations upon imposing suitable differential constraints. It is natural to ask whether we can construct a (2+1)-dimensional extension of this hierarchy which could yield, upon imposing suitable constraints, (2+1)-dimensional integrable systems.

However, there appears to be no straightforward way to include the universal hierarchy into the standard Lie-algebraic R -matrix scheme in spirit of [6, 7, 8, 9, 10, 11] which would enable one to construct the (2+1)-dimensional extension of the hierarchy in question and prove integrability thereof using the central extension approach.

To circumvent this difficulty, we first lift, in spirit of [12, 13, 14], the universal hierarchy (3) to the *cotangent universal hierarchy*, see Eq.(14) below. This *cotangent universal hierarchy* is already amenable to the approach of [7, 8, 9], and integrability of central extensions of (14) follows from the general theory presented there.

In fact, we go even further than that. Motivated by [13, 14], we perform the double central extension of (14) using *two* cocycles rather than one. Commutativity of so constructed flows and integrability of the resulting hierarchy (27) still follows from the general results of the R -matrix scheme of [7, 8, 9]. The first of the cocycles in question, (16), introduces the second space variable y , thus yielding a (2+1)-dimensional rather than (1+1)-dimensional hierarchy, while the second cocycle, (17), introduces dispersion. A quite different way of introducing dispersion that leads to infinite-order differential equations can be found in [15].

What is more, (27) contains a subhierarchy (28) which is precisely the central extension of the original universal hierarchy (3), and integrability of (28) follows from that of (27).

The hierarchy (27) is bi-Hamiltonian, and, quite interestingly, the Poisson brackets (22) of (27) are *not* of the operand type, see e.g. [16, 17, 18, 10] and references therein for the latter, but rather of the same type that occurs in (1+1) dimensions and also for (2+1)-dimensional hydrodynamic-type systems [11], as is explicitly revealed by Examples 1 and 2. The corresponding recursion operators, being the ratios of the Poisson tensors in question, also are not of operand (bilocal) type, and thus the systems arising from (27) do not fall under the scope of the no-go theorem of [19], just like Eq.(1), see [24], and the systems studied in [11, 13, 14].

On a more general note, the hierarchy (27), just like the original universal hierarchy (3), admits plenty of finite-component reductions, as discussed in detail in Section 5. In contrast with [13, 14], where the hierarchies are two-component by construction, the hierarchy (27) admits reductions with arbitrarily large number of dependent variables.

Note that if we assume the Lax operator of (27) to have the form (36), that is $\mathbf{l} = (\lambda + u, c\lambda + v)$, and set $\alpha = -1$, and $t_2 = t$ in the t_2 -flow of the corresponding subhierarchy (28) we obtain the equation

$$u_t = \partial_x^{-1} u_{yy} + uu_y - u_x \partial_x^{-1} u_y \quad (1)$$

possessing a hereditary recursion operator (cf. e.g. [24])

$$\Phi = u_x \partial_x^{-1} - u + \partial_y \partial_x^{-1} \quad (2)$$

which generates the whole subhierarchy in question:

$$u_{t_n} = \Phi^n u_x.$$

In turn, Eq.(1) is equivalent to the (2+1)-dimensional hydrodynamic-type system

$$u_t + v_y + uv_x - vu_x = 0 \quad u_y + v_x = 0,$$

that has recently attracted considerable attention, see [20, 21, 22, 23, 24, 15, 13, 14, 25].

1 Universal hierarchy: definition and some known results

Recall that the universal hierarchy (see [1, 2, 3, 4, 5] and references therein) is a set of commuting flows of the form

$$G_{t_n} = [(\lambda^n G)_+, G] = -[(\lambda^n G)_-, G], \quad n \in \mathbb{N}, \quad (3)$$

where

$$G = 1 + \sum_{i=-\infty}^{-1} g_i \lambda^i$$

or

$$G = \sum_{j=0}^{\infty} g_j \lambda^j.$$

For any formal series $a = \sum_{i \in \mathbb{Z}} a_i \lambda^i$ we set $a_+ = \sum_{i \geq 0} a_i \lambda^i$ and $a_- = \sum_{i < 0} a_i \lambda^i$. The commutator $[\cdot, \cdot]$ in (3) is given by

$$[a, b] = ab_x - ba_x.$$

Eq.(3) is equivalent to a hydrodynamic chain for g_i .

A large class of finite-field reductions of (3) can be obtained upon setting [4]

$$G = \frac{1}{\lambda^N} \prod_{i=1}^N (\gamma_i + \lambda).$$

Then for each $n = 1, \dots, N$ equation (3) yields a system of N coupled equations for the Riemann invariants γ_i , and the general simultaneous solution of these N systems can be found [4] using the results of Ferapontov [26].

Straightforward attempts to construct (2+1)-dimensional generalizations of (3) using the central extension procedure encounter considerable difficulties because we do not have an ad-invariant non-degenerate symmetric bilinear form on the loop algebra $\text{Vect}(\mathbb{S}^1)[[\lambda, \lambda^{-1}]]$ (or on $\text{Vect}(\mathbb{S}^1)$ for that matter, see below for details), and therefore there is no simple way to establish commutativity of the resulting flows. In order to circumvent this problem we first construct the *cotangent universal hierarchy* in the next section, and then consider central extensions of this enlarged hierarchy.

2 Cotangent universal hierarchy from the loop algebra

Let $\text{Vect}(\mathbb{S}^1)$ stand for the Lie algebra of (smooth) vector fields on the circle \mathbb{S}^1 over the field \mathbb{K} , where \mathbb{K} is \mathbb{R} or \mathbb{C} . The elements of $\text{Vect}(\mathbb{S}^1)$ can be identified with smooth functions $a(x)$ of spatial variable $x \in \mathbb{S}^1$, and the commutator reads

$$[a, b] = ab_x - ba_x, \quad (4)$$

where $a, b \in \text{Vect}(\mathbb{S}^1)$. Here and below the subscripts x, y and t denote the respective partial derivatives.

Consider the loop algebra over $\text{Vect}(\mathbb{S}^1)$, i.e., the algebra $\mathfrak{v} = \text{Vect}(\mathbb{S}^1)[[\lambda, \lambda^{-1}]]$ of formal Laurent series in the parameter λ with coefficients from $\text{Vect}(\mathbb{S}^1)$. We can readily extend the commutator (4) to \mathfrak{v} by setting

$$[a, b] \equiv \text{ad}_a b := ab_x - ba_x \quad a, b \in \mathfrak{v},$$

where ad is the adjoint action of \mathfrak{v} on itself. Let $\mathfrak{v}^* = \text{Vect}^*(\mathbb{S}^1)[[\lambda, \lambda^{-1}]]$ be the ‘formal’ dual of \mathfrak{v} . We have the standard pairing of \mathfrak{v} and \mathfrak{v}^* given by the formula

$$\langle u, a \rangle = \int_{\mathbb{S}^1} \text{res}(au) dx \quad u \in \mathfrak{v}^* \quad a \in \mathfrak{v}, \quad (5)$$

where $\text{res} \sum_{i \in \mathbb{Z}} \varphi_i \lambda^i := \varphi_{-1}$. Thus, we can define the coadjoint action of \mathfrak{v} on \mathfrak{v}^* by setting

$$\langle \text{ad}_a^* u, b \rangle \stackrel{\text{def}}{=} -\langle u, \text{ad}_a b \rangle, \quad (6)$$

for any $a, b \in \mathfrak{v}$ and any $u \in \mathfrak{v}^*$. We readily find that $\text{ad}_a^* u = 2a_x u + au_x$.

In analogy with [13] consider the Lie algebra $\mathfrak{w} = \mathfrak{v} \ltimes \mathfrak{v}^*$, the semidirect sum of \mathfrak{v} and \mathfrak{v}^* , with the commutator given by

$$[(a, u), (b, v)] := ([a, b], \text{ad}_a^* v - \text{ad}_b^* u), \quad (7)$$

where $a, b \in \mathfrak{v}$ and $u, v \in \mathfrak{v}^*$.

We have a natural nondegenerate bilinear symmetric product on \mathfrak{w} , namely,

$$((a, u), (b, v))_{\mathfrak{w}} = \langle v, a \rangle + \langle u, b \rangle. \quad (8)$$

This product is of Killing type, i.e., the bilinear form (8) is ad-invariant with respect to the commutator (7).

There are two natural decompositions of \mathfrak{w} into the sum of Lie subalgebras, that is,

$$\mathfrak{w} = \mathfrak{w}_+ \oplus \mathfrak{w}_-, \quad \mathfrak{w}_+ = \left(\sum_{i \geq k} u_i \lambda^i \right), \quad \mathfrak{w}_- = \left(\sum_{i < k} u_i \lambda^i \right)$$

for $k = 0$ and 1 . Thus for $k = 0$ and $k = 1$ we have well-defined classical R -matrices

$$R = \frac{1}{2}(P_+ - P_-) = P_+ - \frac{1}{2} = \frac{1}{2} - P_-, \quad (9)$$

where P_{\pm} are projections onto Lie subalgebras \mathfrak{w}_{\pm} . The transformation $\lambda \mapsto \lambda^{-1}$ maps the case of $k = 0$ into that of $k = 1$, and vice versa. For this reason in what follows we restrict ourselves to considering the case of $k = 0$ only, and hence P_+ and P_- will stand for projections onto nonnegative and negative powers of λ . The R -matrix (9) defines a new commutator on \mathfrak{w} , viz.

$$[\mathbf{u}, \mathbf{v}]_R := [R\mathbf{u}, \mathbf{v}] + [\mathbf{u}, R\mathbf{v}]. \quad (10)$$

Note that R satisfies the modified Yang-Baxter equation, $[R\mathbf{u}, R\mathbf{v}] - R[\mathbf{u}, \mathbf{v}]_R = -\frac{1}{4}[\mathbf{u}, \mathbf{v}]$, and therefore (10) satisfies the Jacobi identity.

In fact, we have an infinite family of classical R -matrices

$$R_n = R\lambda^n, \quad n \in \mathbb{Z} \quad (11)$$

and the corresponding new commutators on \mathfrak{w} read

$$[\mathbf{u}, \mathbf{v}]_{R_n} := [R_n\mathbf{u}, \mathbf{v}] + [\mathbf{u}, R_n\mathbf{v}], \quad \mathbf{u}, \mathbf{v} \in \mathfrak{w}. \quad (12)$$

It is straightforward to verify that the R -matrices (11) and the commutators (12) are well-defined, and (12) satisfy the Jacobi identity because λ^n is a so-called intertwining operator [9], i.e., it satisfies the condition

$$\lambda^n[\mathbf{u}, \mathbf{v}] = [\lambda^n\mathbf{u}, \mathbf{v}] = [\mathbf{u}, \lambda^n\mathbf{v}].$$

The bilinear form (8) is symmetric and nondegenerate, so in what follows we shall identify \mathfrak{w} with its dual \mathfrak{w}^* using this form.

Let \mathbf{l} be an element of \mathfrak{w} , i.e., we have

$$\mathbf{l} = (l_1, l_2) = \sum_{i \in \mathbb{Z}} (u_i, v_i) \lambda^i. \quad (13)$$

Now we can write down the *cotangent universal hierarchy*

$$\mathbf{l}_{t_n} = [R(\lambda^n \mathbf{l}), \mathbf{l}] = [(\lambda^n \mathbf{l})_+, \mathbf{l}], \quad n = 0, 1, 2, \dots \quad (14)$$

Commutativity of the flows (14) for different n readily follows (see e.g. [9]) from the fact that R is an R -matrix.

In the component form we can write (14) as

$$\begin{aligned} (l_1)_{t_n} &= [(\lambda^n l_1)_+, l_1], \\ (l_2)_{t_n} &= \text{ad}_{(\lambda^n l_1)_+}^* l_2 - \text{ad}_{l_1}^* (\lambda^n l_2)_+, \end{aligned} \quad n = 0, 1, 2, \dots, \quad (15)$$

where the commutator $[\cdot, \cdot]$ is given by (4) and the coadjoint action ad^* is given by (6). It is readily seen that (14) contains (3) as a subhierarchy, namely, the first equation of (15) is nothing but the original universal hierarchy (3) with $G = l_1$.

3 Two-cocycles and related Hamiltonian structures

Assume that the elements of \mathfrak{v} (and hence those of \mathfrak{v}^* and of \mathfrak{w}) depend on an additional independent variable $y \in \mathbb{S}^1$, and modify (5) to become

$$\langle u, a \rangle = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \text{res}(au) \, dx dy, \quad u \in \mathfrak{v}^*, \quad a \in \mathfrak{v}.$$

Now in analogy with [13] define the following two-cocycle on \mathfrak{w} :

$$\omega_1((a, u), (b, v)) = \alpha((a, u), (b, v)_y)_{\mathfrak{w}} = \alpha \int_{\mathbb{S}^1 \times \mathbb{S}^1} \text{res}(av_y - bu_y) \, dx dy, \quad (16)$$

where α is an arbitrary constant. The two-cocycle (16) is an analogue of the standard Maurer–Cartan cocycle. It is immediate that the two-cocycle (16) is bilinear and skew-symmetric, and the Jacobi identity for the extended Lie algebra follows from the ad-invariance of (8).

Moreover, we have another two-cocycle on \mathfrak{w} , namely,

$$\omega_2((a, u), (b, v)) = \beta((a, u), (0, b)_{3x})_{\mathfrak{w}} = \beta \int_{\mathbb{S}^1 \times \mathbb{S}^1} \text{res}(ab_{3x}) \, dx dy, \quad (17)$$

where β is an arbitrary constant. The two-cocycle (17) is a generalization of the standard Gelfand–Fuchs cocycle defining the central extension of the Lie algebra $\text{Vect}(\mathbb{S}^1)$ of vector fields on the circle to the well-known Virasoro algebra.

The authors of [13, 14] consider the Euler equation on the central extension of $(\text{Vect}(\mathbb{S}^1) \ltimes \text{Vect}^*(\mathbb{S}^1))$ constructed using the standard Maurer–Cartan and Gelfand–Fuchs cocycles. The Euler equation in question represents a (2+1)-dimensional bi-Hamiltonian system that includes a subsystem (1) studied in [20, 21, 22, 23, 24, 13, 14, 15, 25].

In our approach the first two-cocycle (16) introduces the second spatial variable y , and the two-cocycle (17) produces certain dispersive terms. We consider a double central extension of the Lie algebra \mathfrak{w} defined using the two-cocycles (16) and (17) that generalize those from [13, 14]. The presence of an additional parameter λ in our approach yields a much larger class of (2+1)-dimensional bi-Hamiltonian systems than the approach of [13, 14]. Roughly speaking, the results of [13, 14] can be obtained from the ours upon setting $\lambda = 0$. Below we employ the classical R -matrix scheme for loop algebras, as presented in [6, 7, 8, 9], in order to construct the related integrable systems.

The space $\mathcal{C}^\infty(\mathfrak{w}^* \cong \mathfrak{w})$ consists of functionals $H(\mathbf{l})$, with densities being quasi-local functions in the sense of [27] of the form

$$H(\mathbf{l}) = \int_{\mathbb{S}^1 \times \mathbb{S}^1} h(\dots, \vec{u}_{xy}, \vec{u}_x, \vec{u}_y, \vec{u}, \partial_x^{-1} \vec{u}_y, \partial_y^{-1} \vec{u}_x, \dots) \, dx dy, \quad (18)$$

where \vec{u} includes all fields u_i, v_i from (13) and coefficients are from \mathbb{K} only, i.e., an explicit dependence of coefficients on x or y is not allowed, cf. [7, 8, 9, 27]. The differential dH of an arbitrary functional $H(\mathbf{l}) \in \mathcal{C}^\infty(\mathfrak{w})$ has the form

$$\mathfrak{w} \ni dH = \left(\frac{\delta H}{\delta l_2}, \frac{\delta H}{\delta l_1} \right) = \sum_i \left(\frac{\delta H}{\delta v_i}, \frac{\delta H}{\delta u_i} \right) \lambda^{-i-1}, \quad (19)$$

so that we have

$$(\mathbf{l}_t, dH)_{\mathfrak{w}} = \int_{\mathbb{S}^1 \times \mathbb{S}^1} \sum_i \left((u_i)_t \frac{\delta H}{\delta u_i} + (v_i)_t \frac{\delta H}{\delta v_i} \right) \, dx dy,$$

where $\frac{\delta H}{\delta u_i}, \frac{\delta H}{\delta v_i}$ are (generalized) variational derivatives with respect to the fields u_i, v_i .

The natural Lie–Poisson bracket on the double central extension of $\mathcal{C}^\infty(\mathfrak{w})$ with the two-cocycles (16) and (17) reads

$$\{H, F\}(\mathbf{l}) = (\mathbf{l}, [dF, dH])_{\mathfrak{w}} + \omega_1(dF, dH) + \omega_2(dF, dH), \quad (20)$$

where $\mathbf{l} \in \mathfrak{w}$ and $H, F \in \mathcal{C}^\infty(\mathfrak{w})$. Let $dH = (\delta_2 H, \delta_1 H)$, where $\delta_i H \equiv \frac{\delta H}{\delta l_i}$ for $i = 1, 2$. Then, the related Poisson tensor π such that

$$\{H, F\} = (dF, \pi dH)_{\mathfrak{w}} \quad \pi dH = ((\pi dH)_1, (\pi dH)_2)$$

is given by the formulas

$$\begin{aligned} (\pi dH)_1 &= [\delta_2 H, l_1] + \alpha(\delta_2 H)_y \\ (\pi dH)_2 &= \text{ad}_{\delta_2 H}^* l_2 - \text{ad}_{l_1}^* \delta_1 H + \alpha(\delta_1 H)_y + \beta(\delta_2 H)_{3x}. \end{aligned} \quad (21)$$

Furthermore, the R -brackets (12) induce a family of new Lie–Poisson brackets on $\mathcal{C}^\infty(\mathfrak{w})$, namely,

$$\{H, F\}_n(\mathbf{l}) = (\mathbf{l}, [dF, dH]_{R_n})_{\mathfrak{w}} + \omega_1^{R_n}(dF, dH) + \omega_2^{R_n}(dF, dH), \quad (22)$$

where

$$\omega_i^{R_n}(\mathbf{u}, \mathbf{v}) := \omega_i(R_n \mathbf{u}, \mathbf{v}) + \omega_i(\mathbf{u}, R_n \mathbf{v}), \quad \mathbf{u}, \mathbf{v} \in \mathfrak{w}, \quad i = 1, 2. \quad (23)$$

All quantities (23) are two-cocycles of the respective brackets (12). This follows from the fact that R satisfies the classical Yang–Baxter equations and λ^n are intertwining operators. Hence, (23) yields a well-defined double central extension (22) of the standard Lie–Poisson bracket. The related Poisson tensors π_n such that $\{H, F\}_n = (dF, \pi_n dH)_{\mathfrak{w}}$ and $\pi_n dH = ((\pi_n dH)_1, (\pi_n dH)_2)$ have the form

$$\begin{aligned} (\pi_n dH)_1 &= [R_n \delta_2 H, l_1] + R_n^* [\delta_2 H, l_1] + (R_n + R_n^*) (\alpha(\delta_2 H)_y) \\ (\pi_n dH)_2 &= \text{ad}_{R_n \delta_2 H}^* l_2 - \text{ad}_{l_1}^* R_n \delta_1 H + R_n^* (\text{ad}_{\delta_2 H}^* l_2 - \text{ad}_{l_1}^* \delta_1 H) \\ &\quad + (R_n + R_n^*) (\alpha(\delta_1 H)_y + \beta(\delta_2 H)_{3x}), \end{aligned} \quad (24)$$

where R_n^* is the adjoint of R_n with respect to (8). We readily see that $R_n^* = -\lambda^n R$. It is important to note that all Poisson tensors π_n are pairwise compatible. Let us stress once more that the Poisson structures (22) are not of operand type [16, 17, 18, 10] but rather of the pseudodifferential type more characteristic of (1+1)-dimensional integrable systems.

Now we can write down the explicit form of the double central extension of the cotangent universal hierarchy (14) using the two-cocycles (16) and (17).

4 Lax formalism for the double central extension of cotangent universal hierarchy

The Casimir functionals $C_\iota \in \mathcal{C}^\infty(\mathfrak{w})$ of natural Lie–Poisson bracket (20) are in involution with respect to all Lie–Poisson brackets (22). Taking the functionals C_ι for the Hamiltonians yields multi-Hamiltonian dynamical systems on \mathfrak{w} of the form

$$\mathbf{l}_{t_\iota} = \dots = \pi_{n-1} dC_{\iota+1} = \pi_n dC_\iota = \pi_{n+1} dC_{\iota-1} = \dots \quad (25)$$

Here the functionals $C_\iota, \iota \in \mathbb{Z}$, are such that

$$dC_{\iota+n} = \lambda^n dC_\iota, \quad n \in \mathbb{Z}.$$

The Casimir functionals C_ι are in involution with respect to (22), so the flows (25) commute.

In order to find the explicit form of the dynamical system (25) we need the annihilators Ω_ι of (21) such that $\pi\Omega_\iota = 0$. Hence, $\Omega_\iota = (\Omega_1^\iota, \Omega_2^\iota)$ must satisfy

$$\begin{aligned} 0 &= [\Omega_1^\iota, l_1] + \alpha(\Omega_1^\iota)_y \\ 0 &= \text{ad}_{\Omega_1^\iota}^* l_2 - \text{ad}_{l_1}^* \Omega_2^\iota + \alpha(\Omega_2^\iota)_y + \beta(\Omega_1^\iota)_{3x}. \end{aligned} \quad (26)$$

Now, as $\Omega_\iota = (\Omega_1^\iota, \Omega_2^\iota) = (\delta_2 C_\iota, \delta_1 C_\iota)$, the dynamical system (25) can be written in the Lax form:

$$\begin{aligned} (l_1)_{t_\iota} &= [(\Omega_1^\iota)_+, l_1] + \alpha\partial_y(\Omega_1^\iota)_+ \\ (l_2)_{t_\iota} &= \text{ad}_{(\Omega_1^\iota)_+}^* l_2 - \text{ad}_{l_1}^* (\Omega_2^\iota)_+ + \alpha\partial_y(\Omega_2^\iota)_+ + \beta\partial_x^3(\Omega_1^\iota)_+. \end{aligned} \quad (27)$$

This is the sought-for *double central extension* of the cotangent universal hierarchy written in the Lax form. Commutativity of the flows (27) with different values of n follows from the general theory of the central extension procedure, see [7, 8, 10, 11], upon making use of existence of the ad-invariant nondegenerate bilinear symmetric form (8) on \mathfrak{w} .

We readily see from (26) that Ω_1^ι involves the fields from l_1 only. Hence, the equations for l_1 in (27) form a Lax subhierarchy of the form

$$(l_1)_{t_\iota} = [(\Omega_1^\iota)_+, l_1] + \alpha\partial_y(\Omega_1^\iota)_+. \quad (28)$$

The flows (28) commute because so do their counterparts in (27). The Lax hierarchy (28) is a central extension of the universal hierarchy (3) considered in [2]. This extension involves a new independent variable y .

If $\beta = 0$ then (28) can be obtained from (27) as a reduction under the constraint $l_2 = 0$. However, the Hamiltonian structures given by (22) admit no Dirac reduction in this case, because from the second equation of (22) with $\beta = 0$ for the constraint $l_2 = 0$ we conclude that the terms with $\delta_2 H$ vanish, and one cannot express $\frac{\delta H}{\delta l_2}$ in the terms of $\frac{\delta H}{\delta l_1}$.

5 Finite-field Lax operators

It is somewhat difficult to work with the general Lax operators of the form (13). In this section we consider the elements of \mathfrak{w} being formal Laurent series at infinity and having finite highest orders, that is,

$$\boldsymbol{l}^{(\infty)} = (u_{N_1}\lambda^{N_1} + u_{N_1-1}\lambda^{N_1-1} + \dots, v_{N_2}\lambda^{N_2} + v_{N_2-1}\lambda^{N_2-1} + \dots). \quad (29)$$

A straightforward analysis of the terms on the left- and right-hand side of (27) reveals that the Lax operators (29) yield consistent Lax equations (27) if $N_2 \geq N_1 \geq -1$, the field u_{N_1} for $N_1 \geq 0$ is a nonzero constant and the field v_{N_2} for $N_2 \geq 0$ is a constant (not necessarily nonzero). Likewise, the formal Laurent series at zero

$$\boldsymbol{l}^{(0)} = (\dots + u_{1-m_1}\lambda^{1-m_1} + u_{-m_1}\lambda^{-m_1}, \dots + v_{1-m_2}\lambda^{1-m_2} + v_{-m_2}\lambda^{-m_2}) \in \mathfrak{w} \quad (30)$$

yield consistent Lax equations (27) if $m_2 \geq m_1 \geq 0$.

However, we still need the explicit form of annihilators of (21) that satisfy (26) for the Lax operators (29) and (30), respectively. For the sake of simplicity we shall restrict ourselves to the case of $N_1 = N_2 = N$ and $m_1 = m_2 = m$. Then it is not difficult to show that the respective solutions of (26) should have the form

$$\Omega_{q,k}^\infty = (a_q\lambda^q + a_{q-1}\lambda^{q-1} + \dots, b_{q+k}\lambda^{q+k} + b_{q+k-1}\lambda^{q+k-1} + \dots), \quad (31)$$

where $k \geq 0$, a_q is a nonzero constant, and b_{q+k} is a constant (not necessarily nonzero), and

$$\Omega_{q,k}^0 = (\dots + a_{1-q}\lambda^{1-q} + a_{-q}\lambda^{-q}, \dots + b_{1-q-k}\lambda^{1-q-k} + b_{-q-k}\lambda^{-q-k}). \quad (32)$$

All unknown functions a_i, b_i in (31) and (32) are auxiliary fields that can be found by solving (26) through successive equating to zero the coefficients at the powers of λ . The class of solutions for (31) and (32) is determined by the class of functionals (18), and hence the integration “constants” explicitly depending on x and y are not allowed.

It is readily seen that we can set without loss of generality

$$\Omega_{q,k}^\nu = \lambda^q \Omega_{0,k}^\nu, \quad \nu = \infty, 0.$$

Hence, it suffices to determine the coefficients of the operators (31) and (32) for $q = 0$.

Eqs.(31) and (32) generate the Lax hierarchies (27) of pairwise commuting flows with the Lax operator \mathbf{l} given by (29) and (30). These Lax hierarchies are multi-Hamiltonian with respect to the Poisson structures (22), i.e.,

$$\mathbf{l}_{t_{q,k}} = \dots = \pi_{-1} dH_{q+1,k}^\nu = \pi_0 dH_{q,k}^\nu = \pi_1 dH_{q-1,k}^\nu = \dots, \quad \nu = \infty, 0 \quad k \geq 0. \quad (33)$$

The respective Hamiltonians $H_{q,k}^\nu$ can be reconstructed from (31) and (32) as follows. The differentials of $H_{q,k}$ on the level of algebra \mathfrak{w} are given by (19) with respect to (29) or (30). Thus, $dH_{q,k}^\nu$ are obtained by projecting $\Omega_{q,k}^\nu$ onto the subspace of \mathfrak{w} spanned by (19). Now, the Hamiltonians $H_{q,k}^\nu$ can be recovered using the homotopy formula [28]

$$H_{q,k}^\nu(\mathbf{l}) = \int_0^1 (\mathbf{l}, dH_{q,k}^\nu(\mu \mathbf{l}))_{\mathfrak{w}} d\mu \quad \nu = \infty, 0. \quad (34)$$

As we deal with the restricted Lax operators (29) and (30), the images of the Poisson tensors (24) do not have to span proper subspace of \mathfrak{w} , i.e., the images of (24) do not have to lie in the space spanned by $\mathbf{l}_{t_{q,k}}$.

It is readily checked that the images of the Poisson tensors π_n do span a proper subspace with respect to (29) if $n \leq N$ for $N \neq -1$ or if $n \leq 0$ for $N = -1$, and with respect to (30) if $n \geq -m$. In the remaining cases the Dirac reduction procedure must be invoked.

Ultimately, we are interested in a construction of closed finite-field systems. Thus, we must restrict (13) so that it contains finitely many dynamical fields and yields consistent Lax hierarchies (27). Combining (29) and (30) reveals that in the generic case the appropriate Lax operators have the form

$$\mathbf{l} = (u_N, v_N)\lambda^N + (u_{N-1}, v_{N-1})\lambda^{N-1} + \dots + (u_{1-m}, v_{1-m})\lambda^{1-m} + (u_{-m}, v_{-m})\lambda^{-m}, \quad (35)$$

where $N \geq -1$, $m \geq 0$, u_N is a nonzero constant and v_N is a constant (not necessarily nonzero). For Lax operators (35) the differentials (31) and (32) generate for $\nu = \infty$ and $\nu = 0$ two commuting multi-Hamiltonian Lax hierarchies (33). Now, the Poisson tensors π_n given by (24) form a proper subspace with respect to (35) if $N \geq n \geq -m$ for $N \geq 0$ and if $0 \geq n \geq -m$ for $N = -1$.

Now let us present examples of two-field (2+1)-dimensional bi-Hamiltonian integrable systems.

Example 1 Consider (35) for $N = 1$ and $m = 0$. The resulting Lax operator has the following simple form:

$$\mathbf{l} = (\lambda + u, c\lambda + v), \quad (36)$$

where c is an arbitrary constant. We consider only the case of $k = 0$ in (31). Then, solving (26) for (31) yields

$$\Omega_{0,0}^\infty = (1 + u\lambda^{-1} + \alpha\partial_x^{-1}u_y\lambda^{-2} + A\lambda^{-3} + \dots, d + (v + 2(c-d)u)\lambda^{-1} + B\lambda^{-2} + C\lambda^{-3} + \dots), \quad (37)$$

where

$$\begin{aligned}
A &= \alpha^2 \partial_x^{-2} u_{yy} - 2\alpha \partial_x^{-1} (u u_y) + \alpha u \partial_x^{-1} u_y, \\
B &= \alpha \partial_x^{-1} v_y + \beta u_{xx} + 2(2c - d) \alpha \partial_x^{-1} u_y - 3(c - d) u^2, \\
C &= \alpha^2 \partial_x^{-2} v_{yy} + \alpha \partial_x^{-1} (u v)_y - 2\alpha u \partial_x^{-1} v_y + \alpha v \partial_x^{-1} u_y + 2\alpha \beta u_{xy} - \frac{1}{2} \beta u_x^2 - \beta u u_{xx} \\
&\quad + 2(3c - d) \alpha^2 \partial_x^{-2} u_{yy} - 2(3c - 2d) \alpha \partial_x^{-1} (u u_y) - 2(3c - 2d) \alpha u \partial_x^{-1} u_y + 4(c - d) u^3,
\end{aligned}$$

and d is another arbitrary constant.

Hence, the annihilator (37) generates, through (33), the following hierarchy of commuting flows:

$$\begin{aligned}
\begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} u_x \\ v_x - 2d u_x \end{pmatrix} \\
\begin{pmatrix} u \\ v \end{pmatrix}_{t_1} &= \begin{pmatrix} \alpha u_y \\ \alpha v_y + \beta u_{3x} + 2(c - d) \alpha u_y - 6(c - d) u u_x \end{pmatrix} \\
u_{t_2} &= \alpha^2 \partial_x^{-1} u_{yy} - \alpha u u_y + \alpha u_x \partial_x^{-1} u_y \\
v_{t_2} &= \alpha^2 \partial_x^{-1} v_{yy} + 2\alpha u_y v - \alpha u v_y - 2\alpha u_x \partial_x^{-1} v_y + \alpha v_x \partial_x^{-1} u_y + 2\alpha \beta u_{xxy} - 2\beta u_x u_{xx} - \beta u u_{xxx} \\
&\quad + 2(2c - d) \alpha^2 \partial_x^{-1} u_{yy} - 2(5c - 4d) \alpha u u_y - 4(2c - d) \alpha u_x \partial_x^{-1} u_y + 12(c - d) u^2 u_x \\
&\vdots
\end{aligned}$$

The evolution equations for u form a subhierarchy of the above hierarchy. In particular, it is straightforward to verify that the t_2 -flow of this subhierarchy yields nothing but (1) if we set $t_2 = t$ and $\alpha = -1$.

The above equations are bi-Hamiltonian with respect to the Poisson tensors (24). The latter do not require the Dirac reduction for $n = 0$ and $n = 1$. The differential of a functional H has the form $dH = \left(\frac{\delta H}{\delta v}, \frac{\delta H}{\delta u} \right) \lambda^{-1}$. Thus one obtains

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_q} = \pi_0 dH_q = \pi_1 dH_{q-1} \quad (38)$$

with the Poisson tensors given by

$$\pi_0 = \begin{pmatrix} 0 & \partial_x \\ \partial_x & -2c \partial_x \end{pmatrix} \quad (39)$$

and

$$\pi_1 = \begin{pmatrix} 0 & \partial_x u - 2u \partial_x + \alpha \partial_y \\ -2\partial_x u + u \partial_x + \alpha \partial_y & \beta \partial_x^3 + \partial_x v + v \partial_x \end{pmatrix}. \quad (40)$$

The related Hamiltonians can be constructed from (37) using (34), and we obtain

$$\begin{aligned}
H_{-1} &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} (v + du) dx dy \\
H_0 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} (uv + (c - d) u^2) dx dy \\
H_1 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left(\alpha u \partial_x^{-1} v_y + \frac{1}{2} \beta u u_{xx} + (2c - d) \alpha u \partial_x^{-1} u_y - (c - d) u^3 \right) dx dy \\
H_2 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left(\alpha^2 u \partial_x^{-2} v_{2y} + \alpha u v \partial_x^{-1} u_y - \alpha u^2 \partial_x^{-1} v_y + \alpha \beta u u_{xy} - \frac{1}{4} \beta u^2 u_{2x} \right. \\
&\quad \left. + (3c - d) \alpha^2 u \partial_x^{-2} u_{yy} - (3c - 2d) \alpha u^2 \partial_x^{-1} u_y + (c - d) u^4 \right) dx dy \\
&\vdots
\end{aligned}$$

We do not consider here the hierarchy generated by the solutions $\Omega_{n,k}^0$ of (26), as the resulting equations for its coefficients are too cumbersome.

Unfortunately, the Poisson tensors (39) and (40) admit no further reduction to the u -subhierarchy of (38), and thus the above approach does not yield a bi-Hamiltonian representation for (1), albeit reducing to the u -subhierarchy the *ratio* $\pi_1\pi_0^{-1}$ reproduces the recursion operator (2) for (1) found in [24].

Example 2 The Lax operator (35) for $N = -m = -1$ has the form

$$\mathbf{l} = (u, v)\lambda^{-1}.$$

We again consider only the case of $k = 0$ in (31). Then, solving (26) for (31) yields

$$\Omega_{0,0}^\infty = \left(1 - \frac{1}{\alpha}\partial_y^{-1}u_x\lambda^{-1} + A\lambda^{-2} + \dots, \left(-\frac{1}{\alpha}\partial_y^{-1}v_x + \frac{\beta}{\alpha^2}\partial_y^{-2}u_{4x}\right)\lambda^{-1} + B\lambda^{-2} + \dots\right), \quad (41)$$

where

$$\begin{aligned} A &= -\frac{1}{\alpha^2} \left(\partial_y^{-1}(u\partial_y^{-1}u_{xx}) - \frac{1}{2}(\partial_y^{-1}u_x)^2 \right) \\ B &= -\frac{1}{\alpha^2} \left(\partial_y^{-1}(u\partial_y^{-1}v_{xx}) - 2\partial_y^{-1}(v\partial_y^{-1}u_{xx}) + 2\partial_y^{-1}(u_x\partial_y^{-1}v_x) - \partial_y^{-1}(v_x\partial_y^{-1}u_x) \right) + \frac{\beta}{\alpha^3}(\dots). \end{aligned}$$

Hence, (41) generate the following hierarchy

$$\begin{aligned} \begin{pmatrix} u \\ v \end{pmatrix}_{t_0} &= \begin{pmatrix} u_x \\ v_x \end{pmatrix} \\ u_{t_1} &= \frac{1}{\alpha} (u\partial_y^{-1}u_{xx} - u_x\partial_y^{-1}u_x) \\ v_{t_1} &= \frac{1}{\alpha} (u\partial_y^{-1}v_{xx} - 2v\partial_y^{-1}u_{xx} + 2u_x\partial_y^{-1}v_x - v_x\partial_y^{-1}u_x) \\ &\quad - \frac{\beta}{\alpha^2} (u\partial_y^{-2}u_{5x} + 2u_x\partial_y^{-2}u_{4x}) \\ &\vdots \end{aligned}$$

The above equations again are bi-Hamiltonian with respect to Poisson tensors (24) that do not require the Dirac reduction for $n = -1$ and $n = 0$. The differential of a functional H now has the form $dH = (\frac{\delta H}{\delta v}, \frac{\delta H}{\delta u})$. Thus, we have

$$\begin{pmatrix} u \\ v \end{pmatrix}_{t_q} = \pi_{-1}dH_{q+1} = \pi_0dH_q,$$

where

$$\pi_{-1} = \begin{pmatrix} 0 & -\alpha\partial_y \\ -\alpha\partial_y & -\beta\partial_x^3 \end{pmatrix}, \quad \pi_0 = \begin{pmatrix} 0 & \partial_x u - 2u\partial_x \\ -2\partial_x u + u\partial_x & \partial_x v + v\partial_x \end{pmatrix}.$$

The respective Hamiltonians are

$$\begin{aligned} H_0 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} v \, dx dy \\ H_1 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left(-\frac{1}{\alpha}u\partial_y^{-1}v_x + \frac{1}{2}\frac{\beta}{\alpha^2}u\partial_y^{-2}u_{4x} \right) dx dy \\ H_2 &= \int_{\mathbb{S}^1 \times \mathbb{S}^1} \left(\frac{1}{\alpha^2}v(\partial_y^{-1}u_x)^2 - \frac{1}{\alpha^2}u\partial_y^{-1}u_x\partial_y^{-1}v_x + \frac{\beta}{\alpha^3}(\dots) \right) dx dy \\ &\vdots \end{aligned}$$

For the same reasons as above, we do not write down the hierarchy generated by $\Omega_{n,k}^0$.

Acknowledgments

This research was supported in part by the Ministry of Education, Youth and Sports of the Czech Republic (MŠMT ČR) under grant MSM 4781305904, and under the Polish Ministry of Science and Higher Education Research Grant no. N N202 404933. B.Sz. appreciates the warm hospitality of the Mathematical Institute of Silesian University in Opava, where the present work was initiated. A.S. was also supported by Silesian University in Opava under grant IGS 9/2008.

The authors thank M. Błaszak, B.G. Konopelchenko, and V. Ovsienko for stimulating discussions. A.S. would also like to thank J.D.E. Grant for the discussion of the paper [15]. We thank the referees for useful suggestions.

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